

# Manifolds admitting stable forms

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## Abstract

In this note we give a direct method to classify all stable forms on  $\mathbb{R}^n$  as well as to determine their automorphism groups. We show that in dimension 6,7,8 stable forms coincide with non-degenerate forms. We present necessary conditions and sufficient conditions for a manifold to admit a stable form. We also discuss rich properties of the geometry of such manifolds.

## 1 Introduction

Special geometries defined by a class of differential forms on manifolds are again in the center of interests of geometers. These interests are motivated by the fact that such a setting of special geometries unifies many known geometries as symplectic geometry and geometries with special holonomy [Joyce2000], as well as other geometries arising in the M-theory [GMPW2004], [Tsimpis2005]. A series of papers by Hitchin [Hitchin2000], [Hitchin2001] and his school [Witt2005], etc., opened a new way to these special geometries. Among them they studied geometries associated with certain stable 3-forms in dimensions 6, 7 and 8 (see the definition of a stable form in section 2 after Proposition 2.2.)

To classify the stable forms on  $\mathbb{R}^n$  one could use the classification by Sato and Kimura [S-K1977] of the stable forms on  $\mathbb{C}^n$  (they are partial cases of prehomogeneous spaces) and to find the corresponding real forms of the complex stable forms. It might be possible to define the automorphism groups of stable forms on  $\mathbb{R}^n$  by using the Sato-Kimura result for the complex case, but it might not be straightforward, since in general the structure of algebraic ideals over  $\mathbb{R}$  is more complicated than the structure of algebraic ideals over  $\mathbb{C}$ . (The real problem here is to know how many connected components the isotropy group of a given 3-form does have.) We also have noticed a proof by Witt in [Witt2005] attempting to define the automorphism group of the stable form of PSU(3)-type, but unfortunately this proof is incomplete (see Remark 4.8 below).

In sections 2, 3 we study some properties of stable forms. In section 4 we classify stable

forms on  $\mathbb{R}^n$  and we determine their automorphism groups. Our classification is based on the Djokovic work [Djokovic1983]. In sections 5, 6, 7 we present certain necessary conditions as well as some sufficient conditions for a manifold to admit a stable form. We also discuss the rich structure of manifolds admitting stable forms in sections 5, 6, 8. In particular we show that for  $n = 7$  or  $8$  the tangent bundle of any manifold  $M^n$  which admits a stable 3-form has a canonical structure of a real simple Malcev algebra bundle.

## 2 Multi-symplectic forms and stable forms

We recall that a  $k$ -form  $\gamma$  on a vector space  $V^n$  over a field  $F$  is called **multi-symplectic**, if the following map

$$I_\gamma : V \rightarrow \Lambda^{k-1}(V^n)^* : v \mapsto v \lrcorner \gamma$$

is injective.

Clearly a 2-form is multi-symplectic, if and only if it is symplectic.

A multi-symplectic form is generic in the following sense. For any  $k$ -form  $\gamma$  we can define its **rank**, denoted by  $\rho(\gamma)$ , as the minimal dimension of the subspace  $W \subset V^*$  such that  $\gamma \in \Lambda^k W$ .

**2.1. Lemma.** *A  $k$ -form  $\gamma$  on  $V^n$  is multi-symplectic, if and only if, its rank is  $n$ .*

*Proof.* It is easy to see that if the rank of  $\gamma$  is less than  $n$  then the linear map  $I_\gamma$  has non-trivial kernel. On the other hand, if  $I_\gamma$  has non-trivial kernel, then  $\gamma$  can be represented as a  $k$ -form in the dual space of the kernel. In fact we have that the dimension of kernel of  $I_\gamma$  is equal to  $n - \rho(\gamma)$   $\square$

From now on we shall assume that  $F = \mathbb{C}$  or  $\mathbb{R}$ . In these cases the space  $\Lambda^k(V^n)^*$  has the natural topology induced from  $F$ .

**2.2. Proposition.** *The set of multi-symplectic  $k$ -forms is open and dense in the space of all  $k$ -forms.*

*Proof.* The equation for  $\gamma \in \Lambda^k(V^n)^*$  defining that  $I_\gamma$  has non-trivial kernel is an algebraic equation, so the set of non-multi-symplectic  $k$ -forms is a closed subset in  $\Lambda^k(V^n)^*$ . It is also easy to check that for any  $k$  there exists a multi-symplectic  $k$ -form on  $V^n$ . Hence follows the statement.  $\square$

Clearly the multi-symplecity is invariant under the action of the group  $GL(F^n)$ . We shall say that a  $k$ -form  $\gamma$  is **stable**, if the orbit  $GL(F^n)(\gamma)$  is open in the space  $\Lambda^k(V^n)^*$ . By

Proposition 2.2 the set of multi-symplectic  $k$ -forms has non-trivial intersection with the orbit of any stable form. Hence follows immediately

**2.3. Corollary.** *A stable form is multi-symplectic.*

The converse statement is true for  $k = 2$  or  $k = n - 2$ . If  $k = 3$  and  $n = 7$ ,  $F = \mathbb{R}$ , it is known that there are 8 types of  $GL(\mathbb{R}^7)$ -orbits of multi-symplectic 3-forms but among them there are only two of them are stable.

We say that two forms are equivalent (or of the same type), if they are in the same orbit of  $GL(V^n)$ -action. Clearly a real form is stable, if and only if its complexification is stable. We also know that each complex orbit has a finite number of real forms [B-C1962, Proposition 2.3]. Thus the classification of real stable forms is equivalent to the classification of complex stable forms plus the classification of the real forms of the complex stable forms. The classifications of complex stable forms is a part of the Sato-Kimura classification of prehomogeneous spaces [S-K1977].

### 3 Symmetric bilinear forms associated to a 3-form on $\mathbb{R}^8$ .

In this section we associate to a 3-form  $\omega^3$  on  $\mathbb{R}^8$  several symmetric bilinear forms which are invariants of  $\omega^3$ . We prove that the only non-degenerate 3-forms (see definition below, after formula (3.4)) are stable forms. For each stable form we shall associate a Lie algebra structure on  $\mathbb{R}^8$ .

We denote by  $I$  the following natural isomorphism from  $\mathbb{R}^8 \otimes \Lambda^8(\mathbb{R}^8)^* \rightarrow \Lambda^7(\mathbb{R}^8)^*$ :

$$(3.1) \quad I(v \otimes \theta) = v \lrcorner \theta.$$

Let  $\omega$  be a 3-form on  $\mathbb{R}^8$ . We associate  $\omega$  with a symmetric bilinear map  $S : \mathbb{R}^8 \times \mathbb{R}^8 \rightarrow \mathbb{R}^8 \otimes \Lambda^8(\mathbb{R}^8)^*$  as follows

$$(3.2) \quad S^\omega(v, w) = I^{-1}((v \lrcorner \omega) \wedge (w \lrcorner \omega) \wedge \omega).$$

Equivalently

$$(3.2.a) \quad S^\omega(v, w) = - \sum_{i=1}^8 e_i \otimes ((v \lrcorner \omega) \wedge (w \lrcorner \omega) \wedge \omega \wedge e_i^*)$$

for any basis  $(e_i)$  in  $\mathbb{R}^8$  and its dual basis  $(e_i^*)$ .

For each  $v \in \mathbb{R}^8$  we define a linear map  $L_v^\omega : \mathbb{R}^8 \rightarrow \mathbb{R}^8 \otimes \Lambda^8(\mathbb{R}^8)^*$  by letting the first variable in  $S^\omega$  to be  $v$

$$(3.3) \quad L_v^\omega(w) = S^\omega(v, w).$$

Now we shall define a symmetric linear form  $B^\omega(v, w) : \mathbb{R}^8 \times \mathbb{R}^8 \rightarrow (\Lambda^8(\mathbb{R}^8)^*)^2$  as follows

$$(3.4) \quad B^\omega(v, w) = \text{Tr}(L_v^\omega \circ L_w^\omega) \in (\Lambda^8(\mathbb{R}^8)^*)^2.$$

We say that  $\omega$  is **non-degenerate**, if the reduced trace form  $\langle B^\omega, \rho^2 \rangle$  is non-degenerate, for some choice of  $\rho \in \Lambda^8(\mathbb{R}^8) \setminus \{0\}$ .

Let  $G_\omega$  be the automorphism group of  $\omega$ . Let us consider the component  $G_\omega^+ := G_\omega \cap \text{Gl}^+(\mathbb{R}^8)$ .

**3.5. Proposition.** *The bilinear forms  $S^\omega$  and  $B^\omega$  are  $\text{Gl}(\mathbb{R}^8)$ -equivariant in the following sense. For any  $g \in \text{Gl}(\mathbb{R}^8)$  we have*

$$(3.5.1) \quad S^{g^*(\omega)}(X, Y) = g^*(S^\omega(g^{-1}X, g^{-1}Y)),$$

$$(3.5.2) \quad B^{g^*(\omega)}(X, Y) = g^*(B^\omega(g^{-1}X, g^{-1}Y)).$$

If  $\omega$  is non-degenerate, then the group  $G_\omega^+$  is a subgroup of  $SL(\mathbb{R}^8)$ . The group  $G_\omega$  preserves the reduced trace form  $\langle B^\omega, \rho^2 \rangle$  for any choice of  $\rho \in \Lambda^8(\mathbb{R}^8)$ .

*Proof.* The computation of (3.5.1) and (3.5.2) is straightforward, so we omit them. The symmetric form  $B^\omega(v, w)$  can be considered as a linear map  $B^\omega : (\mathbb{R}^8) \rightarrow (\mathbb{R}^8)^* \otimes (\Lambda^8(\mathbb{R}^8)^*)^2$ . Let us consider the associated linear map

$$(3.5.3) \quad \det(B^\omega) : \Lambda^8(\mathbb{R}^8) \rightarrow \Lambda^8((\mathbb{R}^8)^* \otimes (\Lambda^8(\mathbb{R}^8)^*)^2) = \Lambda^8((\mathbb{R}^8)^*)^{17}.$$

If  $B^\omega$  is non-degenerate, then the map  $\det(B^\omega)$  is not trivial. From (3.5.2) we deduce that the map  $\det B^\omega$  is  $G_\omega^+$ -invariant map. So for any  $g \in G_\omega^+$  we get from (3.5.3)

$$\det g = (\det g^{-1})^{17}.$$

Since  $\det g > 0$  we conclude that  $\det g = 1$ . Now using (3.5.2) we get the last statement immediately.  $\square$

**3.6. Proposition.** *i) The trace form  $B^\omega$  is compatible with the multiplication  $S^\omega$  in the following sense*

$$B^\omega(S^\omega(a, b), c) = B^\omega(a, S^\omega(b, c)).$$

ii) The trace form  $B^\omega$  is non-degenerate, if and only if  $\omega$  is stable.

*Proof.* The first statement follows immediately from the definition. To prove the second statement we observe that if  $\omega_1$  and  $\omega_2$  are the real forms of the same complex 3-form, then their trace forms are also the real forms of the trace form for the complex 3-form (all these bilinear forms  $S^\omega$  and  $B^\omega$  can be defined for any vector space  $V$  over an arbitrary field.) Thus to check how many real 3-forms are non-degenerate we need to check only 22 representatives of 3-forms in the Djokovic classification [Djokovic1983]. Furthermore we know that a non-degenerate 3-form must be multi-symplectic. Thus it suffices to compute the trace form of 13 multi-symplectic 3-forms in tables XI-XXIII in the Djokovic classification. We wrote a program for computing the trace form  $B^\omega$  to run it under Maple. The program is very simple. We denote by  $e_1^* \wedge \cdots \wedge e_8^*$  by  $\theta$ , where  $e_i^*$  are the coordinate 1-forms on  $\mathbb{R}^8$ . We shall use  $\theta$  to make a (reduced) multiplication  $V \times V \rightarrow V$

$$(3.7) \quad (vw \rfloor \theta) = (v \rfloor \omega) \wedge (w \rfloor \omega) \wedge \omega$$

Clearly we have

$$(3.8) \quad S^\omega(v, w) = vw \otimes \theta.$$

We define structure constants  $A_{ij}^k$  by

$$(3.9) \quad e_i e_j = \sum_k A_{ij}^k e_k$$

Then

$$(3.9.a) \quad S^\omega(e_i, e_j) = \sum_k A_{ij}^k e_k \otimes \theta$$

Now let us compute

$$\begin{aligned} B^\omega(e_l, e_m) &= \sum_n (S(e_l, S(e_m, e_n)), e_n^*) \\ &\stackrel{3.2.a}{=} \sum_{k,n} \langle e_k \otimes (e_l \rfloor \omega) \wedge (e_m e_n \rfloor \omega) \wedge \omega \wedge e_k^* \otimes \theta, e_n^* \rangle \\ &= \sum_{n,p} (e_l \rfloor \omega) \wedge A_{mn}^p (e_p \rfloor \omega) \wedge \omega \wedge e_n^* \otimes \theta \\ (3.10) \quad &= \sum_{n,p} A_{lp}^n \cdot A_{m,n}^p \otimes (\theta)^2. \end{aligned}$$

The result is that the only stable forms numerated by XXIIIa, XXIIIb, XXIIIc by Djokovic have non-degenerate trace forms.

Below we shall compute explicitly the reduced multiplication forms as well as the reduced trace forms  $\langle B^{\phi_i}, (\theta^*)^2 \rangle$  for stable forms  $\phi_i$  on  $\mathbb{R}^8$  from the Djokovic classification.

(Form XXIIIa):  $\phi_1 = e^{124} + e^{134} + e^{256} + e^{378} + e^{157} + e^{468}$ .

(Form XXIIIb):  $\phi_2 = e^{135} + e^{245} + e^{146} - e^{236} + e^{127} + e^{348} + e^{678}$ .

(Form XXIIIc):  $\phi_3 = e^{135} - e^{146} + e^{236} + e^{245} + e^{347} + e^{568} + e^{127} + e^{128}$ .

The reduced multiplication table for the form XXIIIa is:

$$\begin{bmatrix} 0 & -e1 & e1 & 3e2 - 3e3 & -3e8 & 0 & -3e6 & 0 \\ -e1 & -2e2 & -2e2 + 2e3 & -e4 & -e5 & -e6 & 2e7 & 2e8 \\ e1 & -2e2 + 2e3 & 2e3 & e4 & -2e5 & -2e6 & e7 & e8 \\ 3e2 - 3e3 & -e4 & e4 & 0 & 0 & 3e7 & 0 & 3e5 \\ -3e8 & -e5 & -2e5 & 0 & 0 & 3e3 & -3e4 & 0 \\ 0 & -e6 & -2e6 & 3e7 & 3e3 & 0 & 0 & 3e1 \\ -3e6 & 2e7 & e7 & 0 & -3e4 & 0 & 0 & -3e2 \\ 0 & 2e8 & e8 & 3e5 & 0 & 3e1 & -3e2 & 0 \end{bmatrix}$$

The reduced trace form for the form XXIIIa is:

$$\begin{bmatrix} 0 & 0 & 0 & -30 & 0 & 0 & 0 & 0 \\ 0 & 20 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 20 & 0 & 0 & 0 & 0 & 0 \\ -30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30 & 0 & 0 \\ 0 & 0 & 0 & 0 & -30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -30 \\ 0 & 0 & 0 & 0 & 0 & 0 & -30 & 0 \end{bmatrix}$$

The reduced multiplication table for the form XXIIIb is:

$$\begin{bmatrix} 6e8 & 0 & -3e6 & 3e5 & -e1 & 3e2 & -3e4 & 0 \\ 0 & 6e8 & -3e5 & -3e6 & -e2 & -3e1 & 3e3 & 0 \\ -3e6 & -3e5 & 6e7 & 0 & -e3 & -3e4 & 0 & 3e2 \\ 3e5 & -3e6 & 0 & 6e7 & -e4 & 3e3 & 0 & -3e1 \\ -e1 & -e2 & -e3 & -e4 & -2e5 & 2e6 & 2e7 & 2e8 \\ 3e2 & -3e1 & -3e4 & 3e3 & 2e6 & -6e5 & 0 & 0 \\ -3e4 & 3e3 & 0 & 0 & 2e7 & 0 & 0 & 3e5 \\ 0 & 0 & 3e2 & -3e1 & 2e8 & 0 & 3e5 & 0 \end{bmatrix}$$

The reduced trace form for the form XXIIIb is:

$$\begin{bmatrix} 0 & 0 & 0 & -60 & 0 & 0 & 0 & 0 \\ 0 & 0 & 60 & 0 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 & 0 & 0 & 0 \\ -60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 30 \\ 0 & 0 & 0 & 0 & 0 & 0 & 30 & 0 \end{bmatrix}$$

The reduced multiplication table for the form XXIIIc is:

$$\begin{bmatrix} 6e7 - 6e8 & 0 & 3e6 & 3e5 & 3e4 & 3e3 & e1 & -e1 \\ 0 & 6e7 - 6e8 & -3e5 & 3e6 & -3e3 & 3e4 & e2 & -e2 \\ 3e6 & -3e5 & 6e8 & 0 & -3e2 & 3e1 & e3 & 2e3 \\ 3e5 & 3e6 & 0 & 6e8 & 3e1 & 3e2 & e4 & 2e4 \\ 3e4 & -3e3 & -3e2 & 3e1 & -6e7 & 0 & -2e5 & -e5 \\ 3e3 & 3e4 & 3e1 & 3e2 & 0 & -6e7 & -2e6 & -e6 \\ e1 & e2 & e3 & e4 & -2e5 & -2e6 & 2e7 & 2e7 - 2e8 \\ -e1 & -e2 & 2e3 & 2e4 & -e5 & -e6 & 2e7 - 2e8 & -2e8 \end{bmatrix}$$

The reduced trace form for the form XXIIIc is:

$$\begin{bmatrix} 60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 60 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 60 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 60 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 20 \end{bmatrix}$$

□

**3.11. Proposition.** *Each stable form  $\phi$  defines a Lie algebra structure  $[\cdot, \cdot]_\phi$  on  $\mathbb{R}^8$  by the following formula*

$$(3.11.1) \quad \langle [X, Y]_\phi, Z \rangle_\phi = \phi(X, Y, Z),$$

where  $\langle \cdot, \cdot \rangle_\phi$  denotes a reduced trace form of  $\phi$ . Moreover the Lie algebra  $[\cdot, \cdot]_{\phi_i}$  is the non-compact real form of  $sl(3, \mathbb{C})$  for  $i = 1, 2$  and the Lie algebra  $[\cdot, \cdot]_{\phi_3}$  is the compact real form of  $sl(3, \mathbb{C})$ .

*Proof.* First we note that the anti-symmetric bracket  $[\cdot, \cdot]_\phi$  satisfies the following invariant property. For each  $g \in GL(\mathbb{R}^8)$  we have

$$(3.12) \quad [X, Y]_{g^*\phi} = g([g^{-1}(X), g^{-1}(Y)])_\phi.$$

Hence if the Jacobi identity holds at a form  $\phi$ , it also holds at any point in the orbit  $GL(\mathbb{R}^8)(\phi)$ , moreover these Lie brackets are equivalent. Secondly we notice that the bracket  $[\cdot, \cdot]_\phi$  can be extended linearly over  $\mathbb{C}$  and this complexification is the anti-symmetric bracket defined by the complexification of the form  $\phi$  according to the same formula (3.11.1). Thus to verify the Jacobi identity for 3 stable forms  $\phi_i, i = 1, 3$ , it suffices to verify for one of them.

Now we have two proofs for Proposition 3.11. In the first way we compute the Lie bracket defined in (3.11.1) by using our explicit formula for the reduced trace forms of one of the real stable form  $\phi$  and arrive at the above conclusion. The second method uses the Cartan



form on the real form  $su(3)$  of the complex Lie algebra  $sl(3, \mathbb{C})$ . First we compute the reduced trace formula for the Cartan form on the algebra  $su(3)$

$$\rho_3(X, Y, Z) = \langle [X, Y], Z \rangle$$

where  $\langle, \rangle$  denotes the Killing form on  $su(3)$ . We use the following explicit expression taken from [Witt2005] for a multiple of the form  $\rho_3$ :

$$(-1/\sqrt{3})^3 \rho_3 = e^{123} + (1/2)(e^{147} - e^{156} + e^{246} + e^{257} + e^{345} - e^{367}) + (\sqrt{3}/2)(e^{845} + e^{867})$$

where  $(e_i)$  are an orthonormal basis in  $su(3)$  and  $e^{ijk}$  denotes the form  $e^i \wedge e^j \wedge e^k$ . A direct computation (also used Maple) gives us the following multiplication table for  $(4/3) \cdot (-1/\sqrt{3})^3 \rho_3$

$$\begin{bmatrix} 2e8 & 0 & 0 & \sqrt{3}e6 & \sqrt{3}e7 & \sqrt{3}e4 & \sqrt{3}e5 & 2e1 \\ 0 & 2e8 & 0 & -\sqrt{3}e7 & \sqrt{3}e6 & \sqrt{3}e5 & -\sqrt{3}e4 & 2e2 \\ 0 & 0 & 2e8 & \sqrt{3}e4 & \sqrt{3}e5 & -\sqrt{3}e6 & -\sqrt{3}e7 & 2e3 \\ \sqrt{3}e6 & -\sqrt{3}e7 & \sqrt{3}e4 & \sqrt{3}e3 - e8 & 0 & \sqrt{3}e1 & -\sqrt{3}e2 & -e4 \\ \sqrt{3}e7 & \sqrt{3}e6 & \sqrt{3}e5 & 0 & \sqrt{3}e3 - e8 & \sqrt{3}e2 & \sqrt{3}e1 & -e5 \\ \sqrt{3}e4 & \sqrt{3}e5 & -\sqrt{3}e6 & \sqrt{3}e1 & \sqrt{3}e2 & -\sqrt{3}e3 - e8 & 0 & -e6 \\ \sqrt{3}e5 & -\sqrt{3}e4 & -\sqrt{3}e7 & -\sqrt{3}e2 & \sqrt{3}e1 & 0 & -\sqrt{3}e3 - e8 & -e7 \\ 2e1 & 2e2 & 2e3 & -e4 & -e5 & -e6 & -e7 & -2e8 \end{bmatrix}$$

and we compute easily from here (also by using Maple) that the reduced trace formula for  $(-1/\sqrt{3})^3 \rho_3$  is equal to  $(45/4) (diag)$ . So the trace formula is a multiple of the Killing form. In particular it is non-degenerate. From Proposition 3.6 we obtain that the Cartan form on  $su(3)$  is stable. Now we observe that the other Cartan forms on  $sl(3, \mathbb{R})$  and on  $su(1, 2)$  are the real forms of the Cartan form on  $sl(3, \mathbb{C})$  so they are also stable.

(There is also another argument to prove that the Cartan forms are stable without using Proposition 3.6 and using the Djokovic classification. We need only to compute their trace form explicitly and notice that the reduced trace form is a multiple of the Killing form. Then we apply the argument we use in the proof of the classification theorem 4.1 to get the dimension of the isotropy group of the Cartan form and hence we get the stability of the Cartan form.)

Once we know that the reduced trace form is a multiple of the Killing form, we get immediately the first statement of Proposition 3.11.

Clearly the Lie algebra  $g$  must lie in the Lie algebra of the stabilizer of the Cartan form. Comparing these Lie algebras with the Lie algebra of the automorphism group of the

stable forms in the Djokovic classification we conclude that the Cartan form on  $su(3)$  is equivalent with the form  $\phi_3$ , the Cartan form on  $sl(3, \mathbb{R})$  is equivalent to the form  $\phi_1$  and the Cartan form on  $su(1, 2)$  is equivalent to the form  $\phi_2$ . This proves the second statement of Proposition 3.11.  $\square$

## 4 Classification of real stable forms.

We observe that the stability of a  $k$ -form is preserved under the Hodge isomorphism  $\Lambda^k(V^n)^* \rightarrow \Lambda^{n-k}(V^n)^*$ . We shall use notation  $e^{12\cdots k}$  for the form  $e^1 \wedge e^2 \wedge \cdots \wedge e^k$ . We also use notation  $G_\gamma$  for the isotropy group of  $\gamma$  under the action of  $Gl(\mathbb{R}^n)$  and by  $g_\gamma$  the Lie algebra of  $G_\gamma$ .

**4.1. Theorem.** *Suppose that  $3 \leq k \leq n - k$ .*

*i) Then a stable  $k$ -form  $\gamma$  on  $\mathbb{R}^n$  exists, if and only if  $k = 3$  and  $6 \leq n \leq 8$ . Furthermore*

*ii) if  $n = 6$ , then  $\gamma$  is equivalent to one of the following forms:*

$\gamma_1 = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6$  with  $G_{\gamma_1} = (SL(\mathbb{R}^3) \times SL(\mathbb{R}^3) \times) \mathbb{Z}_2$ ;

$\gamma_2 = \text{Re}(e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6)$  with  $G_{\gamma_2} = SL(\mathbb{C}^3)$ ,

*iii) if  $n = 7$ , then  $\gamma$  is equivalent to one of the following forms:*

$\omega_1 = e^{123} - e^{145} + e^{167} + e^{246} + e^{257} + e^{347} - e^{356}$  with  $G_{\omega_1} = G_2$ ;

$\omega_2 = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}$  with  $G_{\omega_2} = \tilde{G}_2$ ,

*iv) if  $n = 8$ , then  $\gamma$  is equivalent to one of the following forms:*

$\phi_1 = e^{124} + e^{134} + e^{256} + e^{378} + e^{157} + e^{468}$  with  $G_{\phi_1} = SL(3, \mathbb{R}) \times \mathbb{Z}_2$ ;

$\phi_2 = e^{135} + e^{245} + e^{146} - e^{236} + e^{127} + e^{348} + e^{678}$  with  $g_{\phi_2} = PSU(1, 2) \times \mathbb{Z}_2$ ;

$\phi_3 = e^{135} - e^{146} + e^{236} + e^{245} + e^{347} + e^{568} + e^{127} + e^{128}$  with  $G_{\phi_3} = PSU(3) \times \mathbb{Z}_2$ .

*Proof.* We first show that if  $4 \leq k \leq n - k$  then there is no stable form. It suffices to show that in this case we have

$$(4.2) \quad \dim \Lambda^k(\mathbb{R}^n) \geq n^2 + 1 = \dim(Gl(V^n)) + 1.$$

Clearly we have under the assumption that  $4 \leq k \leq n - k$

$$\dim \Lambda^k(\mathbb{R}^n) \geq \dim \Lambda^4(\mathbb{R}^n).$$

Therefore (2.2) is a consequence of the following equality

$$(4.3) \quad f(n) := n^3 - 6n^2 - 13n - 6 \geq 1, \text{ for } n \geq 8.$$

Since  $f'(n) > 0$  for all  $n \geq 8$  it suffices to check (4.3) for  $n = 8$  which is an easy exercise. To complete the proof of Theorem 4.1.i we need to show that stable 3-forms exist for

$n = 6, 7, 8$  and not for  $n \geq 9$ . But this is an well-known fact for  $n = 6, 7$  and it follows from the classification of 3-forms on  $\mathbb{R}^8$  by Djokovic [Djokovic1983]. To show that there is no stable 3-form in  $\mathbb{R}^n$ , if  $n \geq 9$  we can repeat the argument above to show that in this case  $\dim \Lambda^3(\mathbb{R}^n) > \dim Gl(\mathbb{R}^n)$ .

ii) This classification is already well-known, see [Hitchin2000] for an wonderful treatment.

iii) This classification follows from the list of Bures and Vanzura of multi-symplectic 3-forms in dimension 7 [B-V2003] together with their automorphism groups. The groups  $G_{\omega_i}$  have been first determined by Bryant [Bryant1987].

iv) We shall complete this classification from the last table in [Djokovic1983]. In that table Djokovic supplied us only the Lie algebras  $g_{\phi_i}$ , for  $i = 1, 2, 3$ . We shall recover  $G_{\phi_i}$  from  $g_{\phi_i}$  by using the following lemmas 4.4 and 4.5.

**4.4. Lemma.** *Group  $Gl^+(\mathbb{R}^n)$  acts transitively on the orbit  $Gl(\mathbb{R}^n)(\phi_i)$ , for  $\phi_i$  being one of the forms in Theorem 2.1.iv.*

*Proof.* It suffices to show that the intersection  $G_{\phi_i} \cap Gl^-(\mathbb{R}^8)$  is not empty, where  $Gl^-(\mathbb{R}^8)$  denotes the orientation reserving component of  $Gl(\mathbb{R}^n)$ .

- For  $\phi_1$  this intersection contains the following element  $\sigma_{23} \cdot \sigma_{57} \cdot \sigma_{68} \cdot I_1 \cdot I_4$ . Here  $\sigma_{ij}$  denotes the orientation reserving linear transformation which permutes the basic vectors  $v_i$  and  $v_j$  and leaves all other basic vectors, and  $I_j$  denotes the orientation reserving linear transformation which acts as  $-Id$  on the line  $v_j \otimes \mathbb{R}$  and leaves all other basic vectors.

- For  $\phi_2$  this intersection contains the following element  $\sigma_{12} \cdot \sigma_{34} \cdot I_6 \cdot I_7 \cdot I_8$ .

- For  $\phi_3$  this intersection contains  $\sigma_{34} \cdot \sigma_{56} \cdot I_1 \cdot I_7 \cdot I_8$ . □

**4.5. Lemma.** *Group  $Gl^+_{\phi} := Gl^+(\mathbb{R}^8) \cap G_{\phi}$  is connected for  $\phi_i$  being one of the forms in Theorem 4.1.iv.*

*Proof.* We use the observation obtained in section 3 that all three forms  $\phi_i$  are the Cartan forms

$$\rho(X, Y, Z) = \langle X, [Y, Z] \rangle$$

on the Lie algebra  $sl(3, \mathbb{R})$ ,  $su(1, 2)$  and  $su(3)$ , where  $\langle, \rangle$  denotes the Killing form. Hence follows that

$$(4.6) \quad Aut(g_{\phi_i}) \subset G_{\phi_i}.$$

In Proposition 3.11 we have also defined a way to recover the structure of the corresponding Lie algebra from  $\phi_i$ . Since all the reduced bilinear forms are invariant with respect to  $G_{\phi_i}$

we get

$$(4.7) \quad G_{\phi_i} \subset \text{Aut}(g_{\phi_i}).$$

Finally the structure of  $\text{Aut}(g_{\phi_i})$  is well-known, see e.g. [Murakami1952] and the references therein. Thus we get Lemma 4.5 from (4.6) and (4.7).  $\square$

Actually the proof of Lemma 4.5 implies Lemma 4.4. Nevertheless the proof of Lemma 4.4 gives us explicitly an element in  $Gl^-(\mathbb{R}^8) \cap G_{\phi_i}$ . This completes the proof of Theorem 4.1.  $\square$

**4.8. Remark.** In his thesis [Witt2005] Witt gave a proof that the component  $G_{\phi_3}^+$  is  $PSU(3)$ . His proof is incomplete, since he used implicitly without a proof the fact that the component  $G_{\phi_3}^+$  preserves the Killing metric on  $su(3)$ . (His method is to associate the Cartan form to a bilinear form with value on  $\mathbb{R}^8$  by using a fixed basis of  $\mathbb{R}^8$ . A detailed analysis shows that such a use is equivalent to giving a linear map from  $(\mathbb{R}^8)^*$  to  $\mathbb{R}^8$  and in the given case of Witt, that map is an isomorphism defined by the Killing metric).

We say that a differentiable form  $\gamma$  on a manifold  $M^n$  is stable, if at each  $x \in M$  the form  $\gamma(x)$  is stable.

**4.9. Proposition.** *If a connected manifold  $M^n$  admits a differentiable stable form  $\gamma^3$ , then for all  $x \in M^n$  the form  $\gamma(x)$  has the same type. In particular  $M^n$  admits a  $G_{\gamma(x)}$  structure. Conversely, if  $M^n$  admits a  $G_\gamma$  structure, then it admits a differentiable form of  $\gamma$  type.*

*Proof.* For each  $x \in M^n$  denote by  $U(x)$  the set of all points  $y \in M^n$  such that  $\gamma^3(y)$  has the same type as  $\gamma^3(x)$ . Clearly  $U(x)$  is an open subset in  $M^n$ . Suppose that  $U(x) \neq M^n$ . Then the closure  $\bar{U}(x)$  contains a point  $y$  which is not in  $U(x)$ . Clearly  $\gamma(y)$  also has the same type as  $\gamma(x)$  since  $U(y)$  has a non-empty intersection with  $U(x)$ . Thus  $y \in U(x)$  which is a contradiction. The last statement follows from the fact that the transition functions on  $G(x)$ -manifold preserve the form  $\gamma(x)$ .  $\square$

## 5 Stable 3-forms on 6-manifolds

### 5.1. Obstruction for the existence of a $G_{\gamma_1}$ -structure.

If a non-orientable manifold  $M^6$  admits a  $G_{\gamma_1}$ -structure, then its orientable double covering shall admit  $G_{\gamma_1}$ -structure. Now we shall concern only orientable manifolds  $M^6$  and so only

the identity component of  $G_{\gamma_1}$ . Clearly  $M^6$  admits  $SL(3) \times SL(3)$ -structure, if and only if it admits a distribution of oriented 3-planes on  $M^6$ .

We denote by  $\rho_2 : H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}_2)$  the modulo 2 reduction.

**5.1.1. Proposition.** *Suppose that a closed manifold  $M^6$  admits a  $SL(3) \times SL(3)$ -structure. Then its Euler class vanishes. Assume that  $H^4(M^6, \mathbb{Z})$  has no 2-torsion, the Euler class  $e(M^6)$  vanishes and  $M^6$  satisfies moreover the following condition (P). There are classes  $c_1, c_2 \in H^2(M, \mathbb{Z})$  such that*

$$(P) \quad p_1(M^6) = c_1^2 + c_2^2, \quad \rho_2(c_1 + c_2) = w_2(M^6).$$

*Then  $M^6$  admits a  $SL(3) \times SL(3)$ -structure.*

*Proof.* The first statement is well-known, since the Euler class of an oriented 3-dimensional vector bundle is a 2-torsion, and  $H^6(M, \mathbb{Z})$  has no 2-torsion. Let us assume that an orientable manifold  $M^6$  with vanishing Euler class has no 2-torsion in  $H^4(M, \mathbb{Z})$ , moreover  $M^6$  satisfies condition (P). Let  $V$  be a non-vanishing vector field on  $M^6$ . Since  $M^6$  satisfies condition (P), there is an almost complex structure  $J$  on  $M^6$  such that  $c_1(J) = c_1 + c_2$ , where  $c_1$  and  $c_2$  satisfies condition (P). Let  $W^4$  be a  $J$ -invariant sub-bundle of  $TM^6$  which is complement to  $V$  and  $JV$ . Clearly  $p_1(W^4) = p_1(M^6)$ . Let  $L_1$  and  $L_2$  be the complex line bundles over  $M^6$  with the first Chern classes  $c_1$  and  $c_2$  satisfying condition (P). Then  $p_1(W^4) = p_1(M^6) = p_1(L_1 \oplus L_2)$  and  $w_2(W^4) = w_2(M^6) = w_2(L_1 \oplus L_2)$ . Hence according to [Thomas1967Z, Lemma1]  $W^4$  and  $L_1 \oplus L_2$  are stably isomorphic. Next we compute that

$$e(W^4) = c_2(W^4) = c_2(TM^6, J) = \frac{1}{2}(c_1^2(TM^6, J) - p_1(TM^6)) = c_1 \cdot c_2 = e(L_1 \oplus L_2).$$

Hence, taking into account [Thomas1967Z, Lemma2]  $W^4$  and  $L_1 \oplus L_2$  are isomorphic as real vector bundles. Thus  $TM^6$  is the sum of two 3-dimensional vector bundles.  $\square$

**5.1.2. Remark.** i) In 5.3 we discuss regular maximal non-integrable  $G_{\gamma_1}$ -structures. If a  $G_{\gamma_1}$ -structure is degenerate, but still regular, then it is easy to see that  $M^6$  satisfies the condition (P).

ii) If  $M^6$  admits 3 linearly independent vector fields, then it admits also a  $SL(3) \times SL(3)$ -structure. In [Thomas1967I] Thomas give a necessary and sufficient condition for an orientable 6-manifold to admit 3 linearly independent vector fields, namely  $M^6$  has vanishing Euler class and vanishing Stiefel-Whitney class  $w_4$ .

*5.2. Obstruction for the existence of a  $G_{\gamma_2}$ -structure.*

**5.2.1. Proposition.** *A manifold  $M^6$  admits a  $SL(3, \mathbb{C})$ -structure, if and only if it is orientable and spinnable.*

*Proof.* Clearly a 6-manifold  $M^6$  admits a  $SL(3, \mathbb{C})$  structure, if and only if  $M^6$  admits an almost complex structure of vanishing first Chern class. In particular  $M^6$  must be orientable and spinnable. On the other hand, if  $M^6$  is orientable and spinnable, then  $M^6$  admits  $SL(3, \mathbb{C})$  structure, since it admits an almost complex structure, whose first Chern class is an integral lift of  $w_2$ . Thus the necessary and sufficient condition for  $M^6$  to admit a  $SL(3, \mathbb{C})$ -structure is the vanishing of the Stiefel-Whitney classes  $w_1(M^6)$  and  $w_2(M^6)$ .  $\square$

### 5.3. Maximal non-integrable 3-forms of $\gamma_1$ -type.

Every 3-form  $\gamma_1$  on  $M^6$  defines a pair two oriented transversal 3-distributions  $D_1$  and  $D_2$  together with volume forms on each  $D_i$  as follows. Recall that at every point  $x \in M$  we can write  $\gamma_1 = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6$ . The union  $D_1 \cup D_2$  is defined uniquely as the set of all vectors  $v \in T_x M$  such that  $\text{rank}(v \lrcorner \gamma_1) = 2$ , or equivalently,  $(v \lrcorner \gamma_1)^2 = 0$ . The orientation (the volume form) of  $D_1$  and  $D_2$  is defined by the restriction of  $\gamma_1$  to each distribution  $D_i$ . Conversely, a pair of two transversal oriented 3-distributions  $D_1$  and  $D_2$  on  $M^6$  together with their volume form defines a 3-form of  $\gamma_1$ -type as follows. Let their volume forms be  $\alpha_1$  and  $\alpha_2$  respectively. Now we define  $\gamma_1 = p_1^*(\alpha_1) + p_2^*(\alpha_2)$ , where  $p_1 : TM \rightarrow D_1$  and  $p_2 : TM \rightarrow D_2$  are the projections defined by  $D_i$ .

We call structure  $(M^6, \gamma_1)$  **regular**, if the dimensions of the distributions  $[D_i, D_i]$  defined by  $\gamma_1$  are constant over  $M^6$ . We shall call a regular  $G_{\gamma_1}$ -structure **maximal non-integrable**, if at least one of the distributions  $D_i$  is maximal non-integrable in the sense that  $D_i + [D_i, D_i] = TM$ .

At this place we note that the labeling  $D_1$  and  $D_2$  is well-defined only locally. Globally we may be not able to distinguish, what of the two planes is the  $D_1$ . This ambiguity can be removed, if  $M^6$  is simply connected, since in this case the two line bundles  $\det D_1$  and  $\det D_2$  can be distinguished.

We can describe the maximal non-integrability of  $D_i$  in term of  $\gamma_1$  as follows. Write  $\omega_1 = p_1^*(\alpha_1)$ ,  $\omega_2 = p_2^*(\alpha_2)$ . Locally we can write  $\omega_1 = p_1^*(e^1 \wedge e^2 \wedge e^3)$ ,  $\omega_2 = p_2^*(e^4 \wedge e^5 \wedge e^6)$ .

**5.3.1. Proposition.** *There is a volume form  $D^3\omega_2 \in \Lambda^3(\Lambda^2(D_1))^*$  defined in local coordinates as follows:*

$$D^3(\omega_2) = i_1^*(dp_2^*(e^4) \wedge dp_2^*(e^5) \wedge dp_2^*(e^6)),$$

where  $i_1 : D_i \rightarrow TM$  is the embedding, and  $dp_2^*(e^i)$  are considered as elements of  $(\Lambda^2 TM)^*$ . This expression does not depend on the choice of local 1-forms  $e^i$  considered as 1-forms on  $D_2$ . This volume form is not zero, if and only if  $D_1$  is maximal non-integrable.

*Proof.* We first show that, if  $f^4, f^5, f^6$  is another co-frame in  $D_2$ , so that  $(f^4, f^5, f^6) = g(e^4, e^5, e^6)$  for  $g \in Gl(D_2)$  then

$$(5.3.2) \quad i_1^*(dp_2^*(f^4) \wedge dp_2^*(f^5) \wedge dp_2^*(f^6)) = (\det g) \cdot i_1^*(dp_2^*(e^4) \wedge dp_2^*(e^5) \wedge dp_2^*(e^6)).$$

Proposition 5.3.1 is a local statement, so it suffices to prove on a small disk  $B^7 \subset M^7$ . We denote by  $A$  the open dense subset in the gauge transformation group  $\Gamma(B^7 \times Gl(D_2))$  which is defined by the condition that  $(f^4, e^5, e^6)$  and  $(f^4, f^5, e^6)$  are also a co-frames on  $D_2$ . Then we have  $g = g_3 \circ g_2 \circ g_1$ , where  $g_1$  sends  $(e^4, e^5, e^6)$  to  $(f^4, e^5, e^6)$ ,  $g_2$  sends  $(f^4, e^5, e^6)$  to  $(f^4, f^5, e^6)$  and  $g_3 = g \circ g_1^{-1} \circ g_2^{-1}$ . Now it is straightforward to check (5.3.2) for each  $g_1, g_2, g_3$ . Hence (5.3.2) holds on the open dense set  $A$ . Since the LHS and RHS of (5.3.2) are continuous mappings, the equality (5.3.2) holds on the whole  $Gl(D_2)$ . This proves the first statement. The second statement now follows by direct calculations in local coordinates.  $\square$

Our study of maximal non-integrable  $G_{\gamma_1}$ -structures is motivated by its relation with the parabolic geometry. This structure is a generalization of the famous Cartan 2-distribution in a 5-manifold and it has a canonical conformal structure [Bryant2005]. The Lie algebra of the automorphism group  $Aut(M^6, \gamma_1)$  as well as local invariants of  $(M^6, \gamma_1)$  can be calculated explicitly using the theory of filtered manifolds (see e.g. [Yamaguchi1993].)

## 6 Stable 3-forms on 7-manifolds

### 6.1. Topological conditions for the existence of a stable 3-form on a 7-manifold.

The sufficient and necessary condition for the existence of a  $G_2$ -structure on a 7-manifold  $M^7$  has been established by Gray [Gray1969]. A manifold admits a  $G_2$ -structure, if and only if it is both orientable and spinnable, i.e. the first two Stiefel-Whitney classes vanish.

It has been observed in [Le2007] that a closed 7-manifold  $M^7$  admits a  $\tilde{G}_2$ -structure, if and only if it is orientable and spinnable. The closedness condition originates from the Dupont work on obstructions using the K-theory, but actually what we used is a consequence of Dupont's result, namely the reduction of the  $SO(7)$ -structure on  $M^7$  to a  $SO(3) \times SO(4)$ -structure. Thomas in [Thomas1967M] also proved that any orientable closed spin 7-manifold  $M^7$  admits 3-linear independent vector fields. It seems that we can drop the condition of closedness in the proof of his theorem.

The geometry of  $G_2$ -manifolds has been intensively studied, but the geometry of  $\tilde{G}_2$ -manifolds is barely explored. In [Le2006] we have constructed the first example of a non-homogeneous closed 7-manifold which admits a closed 3-form of  $\tilde{G}_2$ -type.

### 6.2. Malcev algebra structure on 7-manifolds admitting stable 3-forms.

Any stable 3-form  $\phi$  in dimension 7 defines a reduced symmetric bilinear form by the formula [Bryant1987]

$$\langle V, W \rangle_\phi = \langle (V \lrcorner \phi) \wedge (W \lrcorner \phi) \wedge \phi, \rho \rangle$$

where  $\rho$  is some nonzero element in  $\Lambda^8(\mathbb{R}^7)$ . Let us define a multiplication  $x \circ_\phi y$  on  $\mathbb{R}^7$  by the following formula:

$$\langle x \circ_\phi y, z \rangle_\phi = \phi(x, y, z).$$

Peter Nagy explained us that the skew-symmetric multiplication  $x \circ_\phi y$  defines the structure of the simple Malcev algebra  $A^*$  on  $\mathbb{R}^7$  whose corresponding Moufang loop is  $S^7$  for  $\phi = \omega_1$  in Theorem 4.1 (resp. the pseudo sphere  $S_{(4,4)}(1)$  of the unit vector in the vector space  $\mathbb{R}^8$  with the metric with the signature  $(4, 4)$  for  $\phi = \omega_2$ ). Malcev algebras are generalization of Lie algebras, see [Sagle1961] for more information, in particular the structure of the simple Malcev algebras  $A^*$  on  $\mathbb{R}^7$ .

Thus the tangent bundle  $TM^7$  has the canonical structure of the simple Malcev algebra bundle.

## 7 Stable 3-forms on 8-manifolds

As before we assume that  $M^8$  is orientable, since we can go to the orientable double covering, if necessary.

The maximal compact subgroup of  $G_{\phi_1}^+$  is  $SO(3)$  which is included to  $SO(8)$  via the adjoint representation. The maximal compact subgroup of  $PSU(1, 2) = SU(1, 2)/\mathbb{Z}_3$  is  $S(U(1) \times U(2))/\mathbb{Z}_3$ . The subgroups  $SO(3)$  and  $S(U(1) \times U(2))/\mathbb{Z}_3$  are subgroups of  $PSU(3) = SU(3)/\mathbb{Z}_3$ . Thus any orientable 8-manifold  $M^8$  admitting a 3-form of  $\phi_1$ -type or of  $\phi_2$ -type admits also a 3-form of  $\phi_3$ -type. In particular  $M^8$  must be orientable and spinnable. Now for any spinnable manifold  $M^8$  we define the characteristic class  $q_1(M)$  as follows.

Denote by  $q_1$  the spin characteristic class in  $H^4(BSpin(\infty), \mathbb{Z})$  corresponding to  $-c_2 \in H^4(BSU(\infty), \mathbb{Z})$ . For any spin-bundle  $\xi$  over  $M$  we denote by  $q_1(\xi)$  the pull-back of  $q_1$ . We set  $q_1(M) := q_1(TM)$ .

As before  $\rho_2 : H^2(M^8, \mathbb{Z}) \rightarrow H^2(M^8, \mathbb{Z}_2)$  denotes the modulo 2 reduction. The following Proposition is essentially a reformulation of Corollary 6.4 in [CCV2007].

**7.1. Proposition.** *A closed orientable 8-manifold  $M^8$  admits a stable 3-form, if and only if it satisfies the following conditions*

$$(a) \quad w_2(M^8) = 0 = e(M^8),$$



$$(b) \quad w_6(M^8) \in \rho(H^6(M^8, \mathbb{Z})),$$

$$(c) \quad p_2(M^8) = -q_1(M^8)^2 \text{ and } \frac{(q_1(M^8))^2}{9}[M^8] = 0 \pmod{6}.$$

In fact Corollary 6.4.1 in [CCV2007] is formulated as a necessary and sufficient condition for a manifold to admit a  $PSU(3)$ -structure. But we have seen that the necessary condition for a manifold  $M^8$  to admit a  $PSU(3)$ -structure is also a necessary condition for a manifold to admit a  $SL(3, \mathbb{R})$ -structure or a  $PSU(1, 2)$ -structure.

## 8 Further remarks

8.1. It is easy to see that our construction of natural bilinear forms works also for 3-forms on space  $\mathbb{R}^{3n+2}$ . In the same way (this is already noticed first by Bryant for  $\mathbb{R}^7$ , in [B-V2003] this form has been computed for all except one multi-symplectic 3-forms) we can associate to any 3-form  $\omega$  on  $\mathbb{R}^{3n+1}$  a bilinear form with values in  $\Lambda^{3n+1}(\mathbb{R}^{3n+1})^*$ , and it descends to a bilinear form if the 3-form is non-degenerate; we can also associate to any 3-form  $\omega$  on  $\mathbb{R}^{3n}$  a linear map from  $\mathbb{R}^{3n}$  to  $\mathbb{R}^{3n} \otimes \Lambda^{3n}(\mathbb{R}^{3n})^*$ , and this linear map descends to a linear map  $\mathbb{R}^{3n} \rightarrow \mathbb{R}^{3n}$ , if the 3-form  $\omega$  is non-degenerate (this is noticed by Hitchin for  $\mathbb{R}^6$ ). We have not yet tested, if non-degenerate 3-forms exist in higher dimensions. In low dimensions 6,7,8 they coincide with stable forms.

8.2. Let  $\omega^3$  be a stable 3-form on  $M$ ,  $\dim M \geq 7$ . Then there is the canonical inclusion  $G_\omega$  to  $O(k, l)$ . So if a manifold  $M$  admits a stable form  $\omega^3 \neq \gamma_i$ ,  $i = 1, 2$ , it also admits a canonical (pseudo)-Riemannian metric. The curvature of this (pseudo)-Riemannian metric is a differential invariant of manifold  $(M, \omega^3)$ . Using these metrics and existing stable forms we can construct new differential forms which appear in other special geometries. Now we shall call a manifold  $(M, \omega^3)$  stable, if  $\omega^3$  is stable. Stable 8-manifolds  $(M^8, \omega^3)$  seem to us special interesting, since the bundle  $TM^8$  has the canonical commutative multiplication as well as the structure of Lie algebra bundle defined in Proposition 3.11. We conjecture that the algebra  $\mathbb{R}^8$  with the commutative multiplication defined by  $\phi_i$  is a simple algebra. We have a partial proof for that conjecture in the case of  $\phi_3$ . The stable form  $\phi_i$  also defines the volume form on  $M^8$  and therefore according to Djokovic it defines the graded  $E_8$ -structure on the bundle  $\oplus_{i=1}^3 (\Lambda^i(TM) \oplus \Lambda^i(T^*M)) \oplus \text{End}(TM)$ .

8.3. Suppose that  $M$  is a compact manifold and  $\omega^3$  is a stable 3-form on  $M$ . As we have seen from 8.2 if  $\dim M \geq 7$ , then the automorphism group  $\text{Aut}(M, \omega^3)$  is a finite dimensional Lie group. If  $\gamma_1$  is maximal non-integrable, then the automorphism group

$(M^6, \gamma_1)$  is also a finite dimensional Lie group. If  $\gamma_1$  is degenerate, then the automorphism group  $Aut(M^6, \gamma_1)$  can be infinite dimensional. Example is  $M^6 = S^1(\theta^1) \times S^1(\theta^2) \times \Sigma_1 \times \Sigma_2$  and  $\omega^3 = d\theta^1 \wedge \sigma^1 + d\theta^2 \wedge \sigma^2$ , where  $\sigma^i$  is the volume element on the surface  $\Sigma_i$ . Finally the automorphism group  $Aut(M^6, \gamma_2)$  is also finite dimensional, since  $SL(3, \mathbb{C})$  is elliptic.

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